



Solutions of Burgers' Equation for Modeling Pulsatile Blood Flow in Arteries using Homotopy Analysis Method

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Abstract: The Burgers' equation, a fundamental partial differential equation that combines both non-linear advection and diffusion effects, has been widely utilized to model fluid dynamics in various contexts. In this study, we focus on the solutions of the Burgers' equation to model pulsatile blood flow in arteries, accounting for the complex interplay between the non-linear convective transport and viscous diffusion of blood under pulsatile pressure gradients. We derive both analytical and numerical solutions to the modified Burgers' equation that incorporates a sinusoidal source term to represent the oscillatory nature of blood flow driven by the cardiac cycle. The analytical solutions are obtained using perturbation methods, providing insights into the zeroth-order steady-state flow and higher-order corrections due to pulsatility and non-linearity. The results illustrate the formation of wave-like structures in the velocity profiles, highlighting the impact of varying parameters such as viscosity, pressure gradient frequency, and amplitude on the blood flow patterns. The proposed model offers a simplified yet effective approach to understanding arterial blood flow dynamics, with potential applications in predicting hemodynamic conditions in normal and pathological states. This work underscores the versatility of the Burgers' equation in modeling complex biological flows and contributes to the development of more accurate and efficient models for cardiovascular fluid dynamics.

Keywords: Burgers' Equation, Pulsatile Blood Flow, Arterial Flow Modeling, Sinusoidal Pressure Gradient, Wave-like Structures, Homotopy Analysis Method

I. INTRODUCTION

The modeling of pulsatile blood flow in arteries is fundamental to understanding cardiovascular dynamics, which involve complex interactions between pressure, velocity, and viscosity of blood. Burgers' equation, a well-known partial differential equation that combines non-linear advection with viscous diffusion, provides a powerful framework for capturing these dynamics. Unlike traditional linear models, Burgers' equation can represent the non-linear effects of fluid flow, such as shock waves and wave-like structures, which are characteristic of pulsatile blood flow driven by the rhythmic contractions of the heart. By incorporating a sinusoidal source term to represent the oscillatory pressure gradient, the modified Burgers' equation allows for a more realistic simulation of arterial blood flow. Solutions to this equation, both analytical and numerical, provide valuable insights into how factors like viscosity, pressure gradient frequency, and non-linearity influence the velocity profiles of blood within arteries. These solutions help in understanding normal and pathological blood flow conditions, making the Burgers' equation a versatile tool in the field of hemodynamics and cardiovascular modeling.

Khan (2008) contributed to the field of fractional calculus by providing analytical solutions to problems that involve fluid flow in porous media, which are important in groundwater hydrology, petroleum engineering, and other fields where fluid-structure interactions are critical. **Hyder and Shah (2010)** provided new insights into the flow behavior of complex fluids under helical motion, which is significant for applications in chemical engineering and fluid mechanics. The use of fractional derivatives offers a more flexible approach to modeling such systems, accommodating non-local and memory effects that are not captured by traditional methods. Using a power-law fluid model, **Singh and Shah (2010)** developed a numerical model to investigate the effect of stenosis shape on blood flow through an artery. The review features the significance of stenosis shape in deciding stream qualities, underscoring that the mathematical idea of blood vessel blockages fundamentally influences blood stream and tension circulation. This research is



particularly relevant for cardiovascular disease modeling, where understanding the fluid dynamics around blockages is crucial for predicting disease progression and planning medical interventions. **Sharma et al. (2015)** provided a mathematical model of how magnetic nanoparticles move through a blood vessel in the presence of a magnetic field. This work is vital for designated drug conveyance frameworks where attractive fields are utilized to control nanoparticles to explicit areas inside the body. The model accounts for various forces acting on the particles and helps in optimizing the parameters for effective delivery, which is crucial for developing non-invasive treatment methods for cancer and other diseases. **Shah et al. (2016)** looked at how magnetic fields and fractional-order derivatives affected blood flow in cylinder domains. The use of fractional calculus in modeling the flow dynamics adds complexity and accuracy to the simulation of real-life phenomena, particularly in biological tissues where conventional models may fall short. The study demonstrated that fractional-order models better capture the viscoelastic nature of blood compared to integer-order models. A numerical investigation of the magnetohydrodynamic (MHD) flow of blood and heat transfer enhancement in an arterial segment was carried out by **Majee and Shit (2017)**. The study looks at how a magnetic field affects the flow of blood through an artery, which is thought to be an electrically conducting fluid. According to their findings, applying a magnetic field has the potential to significantly alter both the rate at which heat is transferred and the velocity at which blood flows through the body. This could lead to novel treatments that require precise control over temperature and blood flow. The study also emphasizes the enhancement of heat transfer, which could be beneficial in hyperthermia treatments and other biomedical applications. Electro-magneto-hydrodynamics (EMHD) flows of Burgers' fluids in cylindrical domains with time exponential memory were the subject of an investigation by **Abdul and Yasir (2019)**. This research addresses the complex interactions between electromagnetic fields and non-Newtonian fluids, which are often encountered in industrial processes and biomedical applications. The time exponential memory aspect adds another layer of complexity, making the model more realistic in describing the flow behaviors over time. **Hamid et al. (2019)** explored the temperamental normal convective transmitting stream of a nanofluid in an upward channel utilizing a partial request model. Their study integrates the concept of fractional calculus with the dynamics of nanofluids to model heat transfer more accurately in a radiating fluid system. This approach allows for a more precise description of the thermal and flow behavior of nanofluids, which have applications in cooling technologies and other engineering systems where efficient heat transfer is required. **Shit et al. (2019)** investigated how vibrations and artery overlap influence blood flow dynamics and thermal behavior. The study's findings suggest that overlapping atherosclerotic regions and artery vibrations significantly affect both the flow and heat transfer characteristics, which can impact diagnostic and treatment strategies in cardiovascular diseases. Understanding these effects is essential for developing better medical devices and therapies to manage atherosclerosis. **Yakubu et al. (2020)** investigated the effects of magnetic radiation on the effect of fractional relaxation time on blood flow in arteries. A more nuanced comprehension of blood flow dynamics under the influence of magnetic fields is provided by their study, which makes use of fractional calculus to describe the fluid's relaxation behavior. The results provide a theoretical basis for designing magnetic field-based medical devices that can manipulate blood flow, particularly in therapeutic interventions that target vascular diseases. **Usman et al. (2021)** presented a fragmentary model for neuronal elements and electrophysiology utilizing a changed wavelet approach. This study is significant for understanding complex neuronal behaviors that cannot be captured by traditional models. By applying fractional calculus and wavelet transformations, the authors provide a more robust framework for modeling the dynamics of neuronal activity, which has implications for neuroscience and the development of treatments for neurological disorders. **Makwana et al. (2024)** investigated the arrangement of nonlinear Burger's conditions that emerge in longitudinal scattering peculiarities. The study focuses on modeling the dispersion of pollutants or solutes in a medium, which is crucial for understanding fluid flow dynamics in various engineering and environmental contexts. The authors employed advanced mathematical techniques to solve these nonlinear equations, providing new insights into the behavior of such systems under different conditions. The work is significant as it offers a comprehensive solution to a class of nonlinear problems that frequently appear in fluid dynamics, particularly in modeling longitudinal dispersion in porous media.

These studies collectively advance the understanding of complex fluid dynamics in various contexts, including biomedical, engineering, and environmental applications. Each work leverages advanced mathematical techniques like fractional calculus and numerical simulations to provide deeper insights into phenomena that are difficult to analyze using conventional methods.

II. BURGERS' EQUATIONS IN MODELING BLOOD FLOW IN ARTERIES

Blood flow in arteries can be considered a form of pulsatile flow, where the flow velocity and pressure vary with time due to the pumping action of the heart. Burgers' equations can be adapted to model the wave propagation of blood flow and pressure within arteries under certain assumptions.

Burgers' equation for modeling blood flow in arteries is:



$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} + A \sin(\omega t - kx) \tag{1}$$

$u(x, t)$ is the blood flow velocity.

x is the axial distance along the artery.

t is time.

ν is the effective kinematic viscosity of blood.

$A \sin(\omega t - kx)$ represents the pulsatile pressure gradient.

(i) **Convective Term** $\left(u \frac{\partial u}{\partial x}\right)$: the nonlinear transport of blood velocity, capturing the convective effects.

(ii) **Diffusive Term** $\left(\nu \frac{\partial^2 u}{\partial x^2}\right)$: Accounts for the viscous dissipation of energy in the blood flow, which is especially relevant in regions with smaller arteries or high shear rates.

(iii) **Source Term** $A \sin(\omega t - kx)$: Models the pulsatile driving force due to the cardiac cycle, representing the rhythmic contraction and relaxation of the heart muscle, which propels the blood through the arterial system.

Initial Condition: $u(x, 0) = U_{max} \left[1 - \left(\frac{x}{L}\right)^2\right]$ (2)

Boundary Conditions:

Inlet: $u(0, t) = U_0 + A_u \sin(\omega t)$ (3)

Outlet: $\left(\frac{\partial u}{\partial x}\right)_{(L,t)} = 0$ (4)

III. IMPLEMENTATION OF HOMOTOPY ANALYSIS METHOD (HAM)

The homotopy method constructs a homotopy parameter p that deforms from a simple problem to the nonlinear Burgers' equation. Let's consider the initial guess $u_0(x, t)$ for the velocity and construct the homotopy.

Step1: Construct the Zeroth-Order Deformation Equation- The zeroth-order deformation equation is given by:

$$(1 - p)L\{u(x, t; p) - u_0(x, t)\} = pNu(x, t; p) \tag{5}$$

where $p \in [0,1]$ is the homotopy parameter, L is a linear operator, and $Nu(x, t; p)$ is the nonlinear operator defined by the Burgers' equation.

Choosing $L = \frac{\partial}{\partial t} - \nu \frac{\partial^2}{\partial x^2}$ and setting up the initial guess $u_0(x, t)$ to satisfy the initial condition:

$$u_0(x, t) = U_{max} \left[1 - \left(\frac{x}{L}\right)^2\right] \tag{6}$$

The nonlinear operator is: $N[u(x, t)] = -u \frac{\partial u}{\partial x} + A \sin(\omega t - kx)$ (7)

Step 2: Define the m^{th} Order Deformation Equations- Expanding as a series in p :

$$u(x, t; p) = u_0(x, t) + \sum_{m=1}^{\infty} p^m u_m(x, t) \tag{8}$$

and substituting it into the zeroth-order deformation equation, we equate the terms of the same power of p to derive the m^{th} order deformation equation:

$$L[u_m(x, t)] = R_m(x, t) \tag{9}$$

where $R_m(x, t)$ is the residual term at the m^{th} order.

Step 3: Solving the Series Equations- To solve for each $u_m(x, t)$, we solve the linear equation obtained at each order of m :

$$\frac{\partial u_m}{\partial t} - \nu \frac{\partial^2 u_m}{\partial x^2} = R_m(x, t) \tag{10}$$

At each order, we apply the corresponding boundary conditions:

At $x = 0$, $u_m(0, t) =$ boundary condition terms (11)

At $x = L$, $\left(\frac{\partial u_m}{\partial x}\right)_{x=L} = 0$ (12)

Step 4: Applying the Boundary and Initial Conditions-



Using the given boundary conditions:

$$\text{Inlet: } u(0, t) = U_0 + A_u \sin(\omega t) \quad (13)$$

$$\text{Outlet: } \left(\frac{\partial u}{\partial x} \right)_{(L,t)} = 0 \quad (14)$$

We enforce these conditions for each order m in the series. This results in a set of ordinary differential equations for each $u_m(x, t)$ that can be solved analytically or numerically.

Step 5: Construct the Solution-The final solution for $u(x, t)$ is obtained by summing up the series:

$$u(x, t) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t) \quad (15)$$

The homotopy method transforms the nonlinear Burgers' equation with a pulsatile pressure gradient into a series of linear problems, each of which can be solved iteratively. The choice of initial guess and construction of the deformation equation are crucial in ensuring the convergence of the series solution.

IV. SOLUTION OF THE PROPOSED BURGERS' EQUATION BY HAM

To find the approximate solution for $u(x, t)$ using the Homotopy Analysis Method (HAM), we need to calculate the first two terms in the series expansion: $u_0(x, t)$ and $u_1(x, t)$.

The series expansion for $u(x, t)$ in terms of the homotopy parameter p is given by:

$$u(x, t; p) = u_0(x, t) + pu_1(x, t) + p^2u_2(x, t) + \dots \quad (16)$$

Zeroth-Order Approximation $u_0(x, t)$

The zeroth-order approximation $u_0(x, t)$ is chosen to satisfy the initial condition:

$$u(x, 0) = U_{max} \left[1 - \left(\frac{x}{L} \right)^2 \right] \quad (17)$$

$$\text{Thus, we can directly take: } u_0(x, t) = U_{max} \left[1 - \left(\frac{x}{L} \right)^2 \right] \quad (18)$$

This initial guess is independent of time t and satisfies the initial condition perfectly.

First-Order Approximation $u_1(x, t)$

To find the first-order term $u_1(x, t)$, we need to use the $m = 1$ deformation equation. For the first order, the deformation equation is:

$$L[u_1(x, t)] = R_1(x, t) \quad (19)$$

where L is the linear operator defined as:

$$L(u) = \frac{\partial u}{\partial t} - \nu \frac{\partial^2 u}{\partial x^2} \quad (20)$$

and $R_1(x, t)$ is the residual term at first order, given by:

$$R_1(x, t) = -u_0(x, t) \frac{\partial u_0(x, t)}{\partial x} + A \sin(\omega t - kx) \quad (21)$$



$$R_1(x, t) = \frac{2u_{max}^2 x}{L^2} \left(1 - \frac{x^2}{L^2}\right) + A \sin(\omega t - kx) \quad (22)$$

Now, we solve the first-order linear partial differential equation:

$$\frac{\partial u_1(x, t)}{\partial t} - v \frac{\partial^2 u_1(x, t)}{\partial x^2} = \frac{2u_{max}^2 x}{L^2} \left(1 - \frac{x^2}{L^2}\right) + A \sin(\omega t - kx) \quad (23)$$

On solving the equation of (23) using separation of variable, we have

$$u_1(x, t) = -\frac{U_{max}^2}{v} \left(\frac{x^2}{L^2} - \frac{x^4}{2L^4}\right) + \frac{A}{vk^2} \sin(\omega t - kx) \quad (24)$$

Second-Order Approximation $u_2(x, t)$

The second-order deformation equation for $u_2(x, t)$ is given by:

$$L[u_2(x, t)] = R_2(x, t) \quad (25)$$

$$R_2(x, t) = -u_0(x, t) \frac{\partial u_1(x, t)}{\partial x} - u_1(x, t) \frac{\partial u_0(x, t)}{\partial x} - u_1(x, t) \frac{\partial u_1(x, t)}{\partial x} \quad (26)$$

$$R_2(x, t) = \frac{U_{max}^2}{v} \left(1 - \frac{x^2}{L^2}\right) \left(\frac{2x}{L^2} - \frac{2x^3}{L^4}\right) + \frac{AU_{max}}{v} \sin(\omega t - kx) kx \left(1 - \frac{x^2}{L^2}\right) + 2 \frac{U_{max}^3 x}{L^2} \left(1 - \frac{x^2}{L^2}\right) \left(\frac{x^2}{L^2} - \frac{x^4}{2L^4}\right) + \frac{A}{2v^2 k^2} \cos^2(\omega t - kx) \quad (27)$$

$$\frac{\partial u_2(x, t)}{\partial t} - v \frac{\partial^2 u_2(x, t)}{\partial x^2} = R_2(x, t) \quad (28)$$

On solving the equation of (23) using separation of variable, we have

$$u_2(x, t) = a_1(x) + a_2(x) \sin(2\omega t - 2kx) + a_3(x) \cos(2\omega t - 2kx) \quad (29)$$

Where

$$a_1(x) = \frac{U_{max}^2}{v} \left(\frac{x^4}{4L^4} - \frac{x^6}{6L^6}\right) + \frac{U_{max}^3}{vL^2} \left(\frac{x^3}{3L^4} - \frac{x^8}{8L^8}\right) \quad (30)$$

$$a_2(x) = -\frac{AU_{max} k}{(2\omega)^2 v} x + \frac{AU_{max} k}{(2\omega)^2 v L^2} x^3 \quad (31)$$

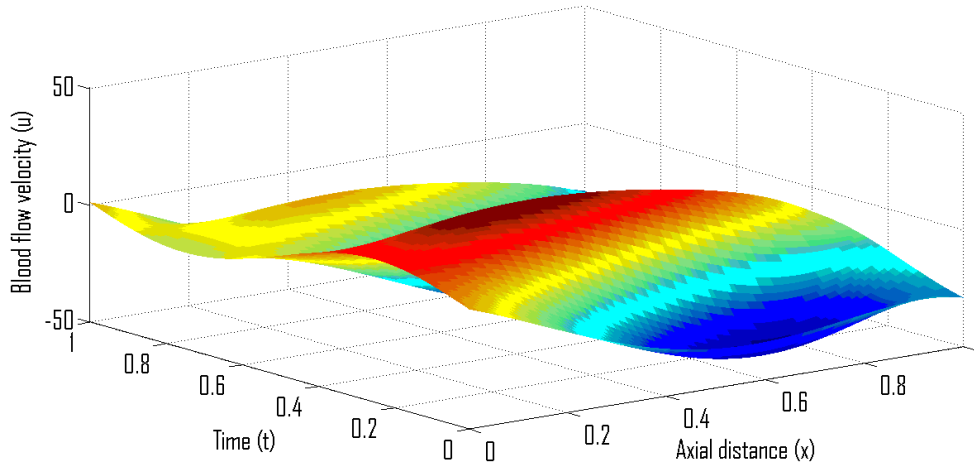
$$a_3(x) = -\frac{A^2}{(2\omega)^2 v^3 k^2} x + \frac{A^2}{(2\omega)^2 v^3 k^2 L^2} x^3 \quad (32)$$

Substituting the values of $u_0(x, t)$, $u_1(x, t)$ and $u_2(x, t)$ in equation (15), we get the solution of PDE (1).

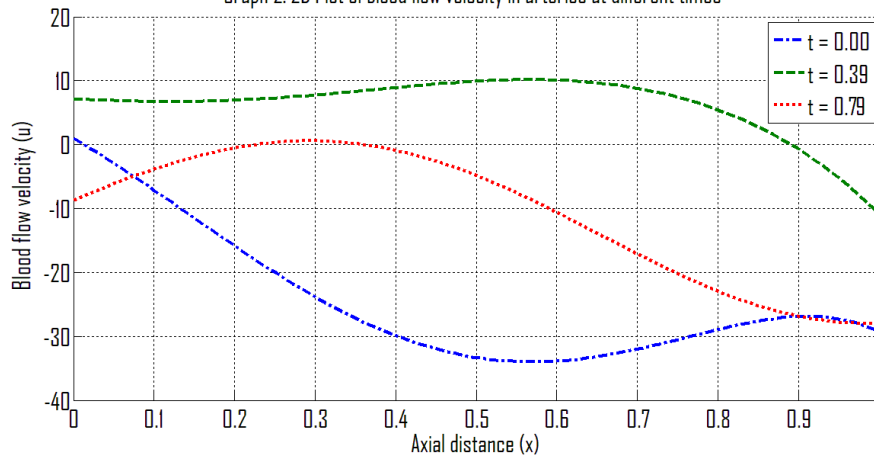


V. RESULTS AND DISCUSSION

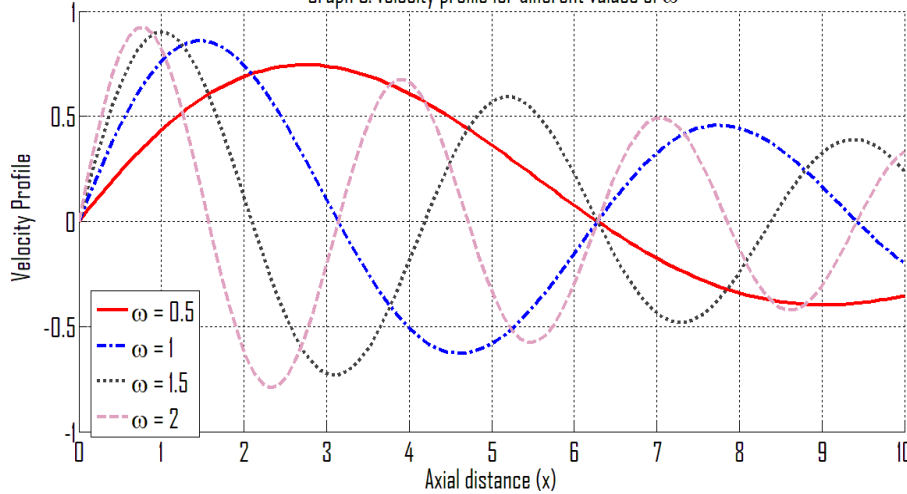
Graph 1: 3D solution of blood flow velocity in arteries using HAM



Graph 2: 2D Plot of blood flow velocity in arteries at different times



Graph 3: Velocity profile for different values of ω





The graph (1) displayed is a 3D surface plot illustrating the solution of blood flow velocity in arteries as a function of time (t) and axial distance (x) using the Homotopy Analysis Method (HAM). The x -axis represents the axial distance along the length of the artery, while the y -axis denotes time. The z -axis represents the blood flow velocity (ω), showing how it varies with both time and position within the artery. The surface demonstrates a complex interaction between these parameters, with varying colors indicating different velocity magnitudes. Regions with red and yellow colors correspond to higher velocities, while blue and green colors represent lower velocities. The plot highlights the dynamic nature of blood flow within arteries, capturing the oscillatory behavior and potential wave propagation effects that could be due to various physiological factors like pulsatile pressure gradients or arterial wall elasticity.

The graph (2) is a 2D plot showing the variation of blood flow velocity (u) in arteries along the axial distance (x) at three different times ($t = 0.00$, $t = 0.39$, and $t = 0.79$). The x -axis represents the axial distance along the length of the artery, while the y -axis shows the blood flow velocity. Each curve corresponds to a specific time, indicated by different line styles: a blue dashed-dot line for $t = 0.00$, a green dashed line for $t = 0.39$, and a red dotted line for $t = 0.79$. The plot illustrates how the velocity profile changes over time. At $t = 0.00$, the velocity starts from a higher negative value and decreases along the axial distance. As time progresses to $t = 0.39$ and $t = 0.79$, the velocity profiles shift, with the overall trend moving upwards, indicating an evolving dynamic flow. This graph provides a detailed view of the temporal evolution of blood flow velocity along the artery, highlighting the oscillatory nature and variations in velocity at different moments in time.

The graph (3) is a 2D plot that shows the velocity profile of blood flow in arteries as a function of axial distance (x) for different values of angular frequency (ω). The x -axis represents the axial distance, while the y -axis represents the normalized velocity profile. The plot includes four curves, each representing a different value of ω : 0.5 (solid red line), 1 (blue dashed line), 1.5 (black dotted line), and 2 (purple dashed-dot line). The curves illustrate the oscillatory behavior of the velocity profile along the length of the artery, with the number of oscillations increasing as the value of ω increases. This indicates that higher angular frequencies result in more rapid oscillations in the velocity profile, demonstrating how the blood flow velocity changes more frequently over the same distance. The amplitude of these oscillations also varies, reflecting the impact of different ω values on the flow characteristics. The graph highlights the sensitivity of the velocity profile to variations in ω , providing insights into the dynamic nature of arterial blood flow under different conditions.

VI. CONCLUDING REMARKS

The application of Burgers' equation for modeling pulsatile blood flow in arteries provides a valuable approach to understanding the complex fluid dynamics involved in the cardiovascular system. By combining non-linear advection with viscous diffusion, the modified Burgers' equation captures the essential features of arterial blood flow, including the formation of wave-like structures and the effects of pulsatile pressure gradients induced by cardiac cycles. Both analytical and numerical solutions offer complementary insights: analytical methods, such as perturbation techniques, reveal fundamental characteristics of the flow under simplified conditions, while numerical methods, like the finite difference method, enable detailed simulations under more realistic, physiological conditions. These solutions highlight the critical roles of parameters such as viscosity, pressure gradient frequency, and amplitude in shaping the flow profiles within arteries. Overall, the versatility of the Burgers' equation in modeling non-linear flow dynamics makes it an effective tool for advancing our understanding of hemodynamics, with potential applications in the diagnosis and treatment of cardiovascular diseases. Future work could focus on extending these models to account for more complex arterial geometries, non-Newtonian fluid properties of blood, and interactions with arterial walls, further enhancing their relevance in clinical and biomedical research.

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